Communications

Counting in threes: Lila's amazing discovery

MELLONY GRAVEN, STEVE LERMAN

On a Saturday in September 2012, Mellony's five year old (5¾ years) daughter, Lila Rose, came running excitedly into the kitchen with the television remote control in hand yelling repeatedly, "Mommy, I've worked out how to count in threes!" Interested in her excitement and its ability to get her away from the television Mellony asked her how she had figured this out. "Look mommy," she said, pointing to the buttons on the remote, "It's 3, 6, 9." Mellony quickly grabbed her cell phone and asked her if she would explain to her again. What follows is the transcript of the subsequent interaction.

Lila: I am going to count in threes look? [*Lila holds up the remote towards the camera*]

Mellony: But how did you work that out show me?

Lila: Cause everyone is 3. [*Lila lifts the remote up and starts pointing with her finger from left to right across the rows of buttons 1 2 3; 4 5 6; 7 8 9—see Figure 1*]

Mellony: So show me how you are counting?

Lila: Three, six, nine, twelve. [*Lila holds her three fingers over each of the number sets 1 2 3; 4 5 6; 7 8 9, then over the 3 buttons below the 789 as she calls out 3; 6; 9; 12*]

Figure 1. Remote buttons.

Mellony: And how did you know that's 12 my angel?

Lila: I don't know, because that's 10 [*Lila points to the unnumbered button under the 7*] and that's 11 [*points to the unnumbered button under the 8*] and that is 12 [*points to the unnumbered button under the 9*]

Mellony: And why do you think it is called counting in threes?

Lila: Because everyone you count in one threes.

Mellony: Show me.

Lila: Three, six, nine, twelve. [*Lila holds her fingers over each row of three buttons as she calls out 3; 6; 9; 12*]

Mellony: And what do you think comes after 12 if we are counting in threes?

Lila: Um, um 13, 13! [*Lila looks to the side, away from the remote, moving her lips mouthing the words one, two, three, … twelve, then aloud*] 13, 13!

Mellony: 13 is the number after 12. That is right, but that is one number after 12, what if you were counting in threes?

Lila: I'm not sure. [*Lila looks at her mom and puts her finger in her mouth seemingly thinking*]

Mellony: Have a look on the buttons [*Lila lifts remote up again and looks at the buttons*] and see if they can help you.

Lila: [*Lila, looking at her mom indicating to her with pointing that there are no more buttons*] There aren't any. Only these. [*She points to the bottom row of three non-numbered buttons*]

Mellony: Can we pretend that there are three maybe?

Lila: [*Looking at the remote and placing her three middle fingers over the four rows of buttons 1 2 3; 4 5 6; 7 8 9; * * * and as she does this says:*] Three, six, nine, 12 13 14 15. 15! [*15 declaratively with a huge smile*]

Mellony: My word my angel, that is brilliant! And what do you think might come after 15?

Lila: [*Rubs her eye and looks down at the remote again counting but this time pointing to the end number in each row only while counting*] Three, six, nine, 12, 15, 18. 18!

Lila was in preschool, in the reception year before formal schooling. When asked about whether she had done this type of counting at school, Lila explained that she had counted in 2s, but never in threes. The teachers at her school explained that she had some experience of counting blocks in pairs (encouraged by a paired arrangement) and some experience of putting counters on every second block on a number chart.

The episode begins with Lila running in to share her discovery with her mother. Two possible motivators for this are her seeking affirmation for her discovery and initiating an opportunity for sharing her finding. Indeed, Lila is a child who enjoys verbally articulating her thinking and enjoys and seeks positive affirmation. Mellony responds in her typical affirming way, similar to if Lila showed her a picture she had drawn, although with added delight and surprise at the mathematical discovery. Recognising the opportunity for extending Lila's discovery combined with Mellony's sometimes teacherish [1] style of engaging with Lila leads to an extended dialogue between mother and child, each catching the other's attention with constant reference to the artefact (the remote) which stimulated the discovery.

Lila is very familiar with the remote having used it to change television channels at least since she was three. On this morning she had begun her day by watching television and would have entered the numbered buttons 3, 0, 5, in order to get her favourite channel. The remote had a range of buttons with words and letters and the numbers 1 to 9 structured in 3 rows of three with three non-numbered coloured buttons below (as shown in Figure 1).

Thus the organisation of the numbers into rows of three had the mathematical notion of groups of three embedded in it:

1	2	3
4	5	6
7	8	9
O	O	O

Lila's explanation involves noticing the 3, 6, 9 column and realising that each of these numbers indicates the number of buttons counting from left to right and top to bottom. The three buttons below the numbers, which are used to extend Lila's discovery by counting up to 12 are labelled "menu", "0" and "help". The numbers on a telephone or lift might similarly afford the notion of counting in groups of threes (or twos, *etc.*) and a similar "counting in" could be noticed by children in such contexts.

So what is going on here? How can we explain Lila's insight, her flash of inspiration? What theoretical tools will help us identify the process of her discovery in a manner that might help us explain what is happening in other similar situations (is the little boy's response in Duckworth, 1972, another instance of the same kind of event? [2]) There is an artefact, a tool; there is a child playing with the artefact/tool and suddenly she sees a number pattern on the buttons that catches her attention and gives her insight into the structure of numbers. There is nothing *specific* in her schooling up to that moment or in any intentional input from her parents that she might have been picking up or repeating.

We have been working with the notion of the zone of proximal development (ZPD) to analyse classroom data and we use this concept to explain Lila's discovery. Our intention in this short piece is to share this explanation and to invite comment and alternative explanations.

Lila, Mellony and the ZPD

Within newer conceptualisations of the ZPD as bi-directional and collaborative (Goos, Galbraith & Renshaw, 2002, p. 196) both mother and child have knowledge but require the other to move it. In Holzman's (1997) terms we see the emergence of a "life space that is inseparable from the we who produce it" (p. 61). From this perspective, several shifts can be noted during the episode. Initially Lila is the activator of the emergence of a ZPD. In this respect, and given that she is holding the artefact which mediated her discovery, Lila is the more knowledgeable other of the discovery and her mother is the learner connecting the relationship between the structuring of the numbered buttons on the artefact and Lila's counting in threes. Once Lila has shown and explained her discovery Mellony becomes the more knowledgeable other, affirming Lila's discovery and confirming her correctness of naming the arrangement as counting in threes. She then asks a series of questions that catch Lila's attention and extend the conversation with continued reference to performing actions (even if imagined) on the remote Lila holds.

In terms of the relationship between artefacts, tasks, talk and social relations (the four parameters of numeracy events described by Askew, Denvir, Rhodes & Brown, 2000), we might describe the episode as: perception of the artefact (the remote) by Lila; noticing and action on artefact by the child leading to the emergence of a "task"—counting in threes; the articulation (*i.e.,* talk) of perceptions and actions through sharing with another (Mellony); extended engagement between Lila and Mellony about task and tool; and abstracted imagined noticing; with the social relations between mother and child influencing all of the above.

From Davydov (1988), we have the idea that the ZPD does not exist prior to a learning activity. In what sense, however, was this event a learning activity? There was no teacher or informed peer, just the artefact. We might call this a self-generated learning activity. When more mature students sit alone and work from a mathematics textbook, the teacher or more informed peer is the absent author of the textbook, and the student may be motivated by impending exams, clarification of ideas from a lecture, or whatever. We might think of the remote as a kind of textbook, the absent author being the designer of the remote face. Lila did not choose to "study" the remote in the scholarly sense of studying for the purpose of learning, however, so the activities are not the same, but there may be some similarities.

From Meira and Lerman (2009), the ZPD emerges, or not, when the participants in the interaction catch each other's attention. We want to suggest that Lila's attention was caught by the layout of the numbers on the buttons, and a ZPD emerged. Using gestures such as holding three fingers over the rows of buttons first on the top row (1, 2, 3), then the second and third rows, and then on the three buttons below the numbers that performed other functions as a continuation, but also at another moment pointing to 3, then 6 below it, then 9 below that, and finally the stand-in button for 12, Lila demonstrated a clear idea that the button arrangement showed how one could count in threes. The ZPD was sustained because she rushed to her mother to show her what she had found and her delighted mother pursued her discovery and, as mathematics teachers would, pushed her further, beyond the buttons with numbers and the three buttons below that could be taken as numbers and beyond, to 15 and 18.

We end by asking whether our account of a this incident extends in some small way the notion of ZPD into a new area that might resonate with experiences that colleagues have encountered, or whether current alternative learning theories might have useful things to say by way of explanation. We also want to consider whether there is anything from this episode that we can use as teachers. Mellony has already used the video clip to indicate to early years teachers that children, when they enter school, may know a lot more about mathematics than the teachers might think, and that their environments (even the TV room) are rich contexts for mathematical exploration and extension if parents choose to engage children in these contexts. How might we structure numeracy lessons to stimulate the emergence of such learning events in contexts where there is increasing prescription of what must be taught and when? Are there other insights that the incident might illuminate for teachers?

Notes

[1] Mellony's teacherish style relates to her identity as a mathematics educator. She taught mathematics for several years, is a mathematics teacher educator and runs a weekly mathematics club.

[2] Duckworth's paper tells of 7 year-old Kevin who, before being told the aim of an activity to put a set of different length drinking straws into order from smallest to biggest, says "I know what I am going to do" and proceeds to take the straws and order them by size himself. He was very proud of himself and Duckworth puts this down to the task having been self-set, as she calls it.

References

Askew, M., Denvir, H., Rhodes, V. & Brown, M. (2000) Numeracy practices in primary schools: towards a theoretical framework. *Research in Mathematics Education* **2**(1), 63–76.

Davydov, V. V. (1988) Problems of developmental teaching. *Soviet Education* **30**, 6-97.

Duckworth, E. (1972) The having of wonderful ideas. *Harvard Educational Review* **42**(2), 217-231.

Goos, M., Galbraith, P. & Renshaw, P. (2002) Socially mediated metacognition: creating collaborative zones of proximal development in small group problem solving. *Educational Studies in Mathematics* **49**(2), 193–223.

Holzman, L. (1997) Schools for growth: radical alternatives to current educational models. Mahwah, NJ: Lawrence Erlbaum Associates.

Meira, L. & Lerman, S. (2009) Zones of proximal development as fields for communication and dialogue. In Lightfoot, C. & Lyra, M. C. D. P. (Eds.) *Challenges and Strategies for Studying Human Development in Cultural Contexts,* pp. 199-219. Rome, Italy: Firera Publishing.

Intercultural dialogue and the geography and history of thought

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First vignette: China and Italy

Mariolina (Bartolini Bussi) is talking at a conference with a Chinese colleague, Xuhua. She has just presented a report on fractions and is writing on the whiteboard with a felt pen. Suddenly Mariolina notices that Xuhua writes fractions in a strange order, first the denominator, then the fraction bar and eventually the numerator:

Mariolina: Why do you write fractions in this way?

Xuhua: What do you mean? How should I write them?

Mariolina: I mean the order. We write them in the reverse order (top-down): first the numerator then the fraction bar and last the denominator.

Xuhua: Very strange, indeed! How do you know how many pieces you wish, if you do not know in how many pieces you have cut the whole?

Second vignette: Italy and Burma (Myanmar)

Mariolina and Alessandro (Ramploud) are talking with two Burmese colleagues (Thein Lwin, a mathematician, and Ko Ta, a doctor and coordinator of a network of Monastic schools) who are visiting their department:

Mariolina: How do you write fractions in Burmese? For instance two thirds.

Thein Lwin: [*Is a bit surprised, writes 2/3 top-down*] Why?

Mariolina: I have read in Wikipedia that the Burmese order is the same as the Chinese one: bottom-up.

Thein Lwin: [*Shakes his head*] No, it's the same as yours!

Ko Ta: [*Smiles*] I am not a mathematician!

Ko Ta closes his eyes, takes a pencil and traces gestures in the air. Alessandro has the impression that Ko Ta is looking for a kind of motion memory of the gesture used when he was a child in a primary school. After some seconds, Ko Ta smiles and shows a bottom-up process: first 3, then the fraction bar and eventually 2.

Thein Lwin: [*Smiles and nods*] He's right. I agree!

These two vignettes tell us a simple story. Chinese and Burmese are in the same family of Sino-Tibetan languages. Hence, it is not surprising that their way of saying fractions (and the process of writing fractions) are similar. Yet in Chinese the traditional process of writing (order) and saying fractions is still the same as in the past, taught in the same way in textbooks, whilst in Burmese it seems that a "Western" habit is changing the tradition. It would be interesting to know whether this process depends on the effect of colonialism (that for decades designed the Burmese education system according to the British tradition) or on the effort to run after Western mathematics and mathematics education as a way to overcome the negative effects of military rule. This issue deserves further analysis; however, it helped the participants in the interaction to reflect on each other's own *un-thought*. Here we are quoting

Jullien (2006), the French philosopher and sinologist, who explains his decision to start to study Chinese and to move to Beijing as a way better to understand the European and Greek philosophy. To observe one's own culture from a distance helps to understand one's own un-thought. The *geography of thought* (Nisbett, 2003) allows us to become aware that our beliefs are relative and that they could have been different had we come from different parts of the world (Bartolini Bussi & Martignone, 2013; Bartolini Bussi *et al.,* 2013).

The history of thought

These stories also raise our curiosity to learn about the *history of thought.* All European languages now share the top-down writing process of fractions and the consequent naming order. What are the roots of this process? *Liber Abaci* (by Leonardo Fibonacci, who introduced the so-called Indo-Arabic notation to Europe), wrote:

> When above any number a line is drawn, and above that is written any other number, the superior number stands for the part or parts of the inferior number; the inferior is called the "denominatus" (denominator), the superior the "denominans" (numerator). Thus, if above the 2 a line is drawn, and above that unity [1] is written, this unity stands for one part of two parts of an integer, *i.e.* for a half, thus ½. (As quoted in Cajori, 1928, p. 269)

Hence we know that the order of describing fractions (and probably, we assume, also that of writing fractions) for Leonardo Fibonacci was (in line with the "Eastern" order):

> denominator → fraction bar → numerator.

Probably the reverse top-down order used later was an effect of the standard way of writing from the top to the bottom of the sheet. The final written products are the same!

Yet there is still the issue of ordinal numbers. Why is the denominator expressed in ordinal numbers? This is even more counter-intuitive. We have not yet found any satisfactory answer to this second question in the books on the history of mathematics or in conversations with historians. We guess that it is related to the importance (as it was already in ancient Egypt) of unit fractions that were used more often than other fractions, and, in some cases, instead of other fractions. There were rules (also studied by Leonardo Fibonacci) that allowed the writing of any fraction as the sum of unitary fractions and this writing helped to solve practical problems in a very effective way. For instance, to divide 5 pizzas among 8 children, one can say that each child has 5/8 of a pizza, but this requires cutting each pizza into 8 pieces and giving 5 pieces to each child. It is quite different from what somebody would do in everyday life! The sum

$$\frac{5}{8}=\frac{1}{2}+\frac{1}{8}$$

mirrors the more natural idea of cutting 4 pizzas in half (to give one half to each child) and then dividing the last one into 8 parts, to give a small piece more to each child. This solution is similar to the one found in ancient civilizations and in the Liber Abaci itself. The recourse to the sequence of unitary fractions in problem solving could have been so natural and frequent that they were considered a special genre of numbers, similar to the natural ones:

$$1,\frac{1}{2},\frac{1}{3},\frac{1}{4},\frac{1}{5},\frac{1}{6}$$

and so on. In this sequence, the order corresponds to the wording of the denominators (at least from the third one). We know that the systematic approach to general fractions with any numerator is a recent idea. Even more recent is the idea of considering fractions in mathematics education as numbers to be represented on a number line, exactly like the whole numbers.

Implications for mathematics education

The case of fractions is just one example of the richness of taking a different perspective on our own un-thought about a mathematical process. Discovering that some issues that had been considered obvious are, on the contrary, the products of long and complex cultural processes prompts teachers to reflect on their beliefs and on the hidden choices made in their context. Although a direct transposition might be impossible, we know that Western languages and traditions are not always the best ones to hint at the genesis of some mathematical processes. In the case of fractions, some Eastern languages seem to be to be facilitators for the construction of meanings (see Siegler *et al.,* 2013).

> **Third vignette: Italy—interaction between an expert and a low achiever**
>
> Anna (Baccaglini-Frank) is working with a low achiever, L, using the software *Motion Math* [1] an app for the iPad, in which learners have to tilt the device to make a falling ball containing a fraction fall towards the right point on the number line [0,1] (for a video, see [2]).
> L seems to be confused by the task. Without an intuition about the position of the fraction on the line it is not easy at all to tilt the device quickly enough during the very short falling time. Anna tries to help him by reading the falling fraction. She is using the Western mode: two thirds, three fourths, and so on.
>
> *Anna:* [*suddenly changes the way of reading*] Let's name the fractions as Chinese do!
>
> *Anna:* [*½ falls*] Of two parts, take one!
>
> *Anna:* [*¾ falls*] Four parts, three!
>
> L is a bit surprised, starts to be less anxious and improves very quickly his performance. The improvement is more evident with unitary fractions (*e.g.,* 1/5).
>
> *L:* Oh yeah, I have to divide the segment into 5!
>
> The same happens with other low achievers.

Motion Math exploits both epistemological and cognitive analyses of fractions (Riconscente, 2013), emphasizing, on the one hand, the importance of using the number line to give coherence to the study of fractions and of whole numbers and, on the other hand, the neurological evidence of the mental number line (Zorzi *et al.*, 2002). Moreover, *Motion Math* exploits embodied learning and, in particular, the integrated perceptual-motor approach (Nemirovsky *et al.,* 2012) in the development of such a mental number line.

From her research on students with mathematics learning difficulties (Karagiannakis *et al.,* in press), and in particular when engaging in interventions with low achievers, Anna is learning to combine neuroscientific findings with the outcomes of the intercultural semiotic analysis discussed in our research group, to smooth the scarce transparency of the Italian wording.

This very short episode from a study in progress shows the synergy between intercultural dialogue, neuroscience and technology for defining effective teaching-learning situations. We hope that this synergy will be further and more deeply developed in the future, and applied in mathematics teacher education and development.

Notes

[1] motionmathgames.com
[2] www.youtube.com/watch?v=hmm0D90vcYI

References

Bartolini Bussi M. G. & Martignone F. (2013) Cultural issues in the communication of research on mathematics education. *For the Learning of Mathematics* **33**(1), 2-8.

Bartolini Bussi M. G., Sun, X. & Ramploud, A. (2013) A dialogue between cultures about task design for primary school. In Margolinas, C. (Ed.) *Task Design in Mathematics Education: Proceedings of ICMI Study* **22**, pp. 549-557. Oxford, UK: ICMI.

Cajori F. (1928) *A History of Mathematical Notation.* London, UK: Open Court.

Jullien F. (2006) *Si parler va sans dire : Du logos et d'autres ressources,* Paris, France : Seuil.

Karagiannakis, G., Baccaglini-Frank, A. & Papadatos, Y. (2014) Mathematical learning difficulties subtypes classification. *Frontiers in Human Neuroscience* **8**(Jan), 57.

Nemirovsky, R., Rasmussen, C., Sweeney, G. & Wawro, M. (2012) When the classroom floor becomes the complex plane: addition and multiplication as ways of bodily navigation. *Journal of the Learning Sciences* **21**(2), 287-323.

Nisbett, R. E. (2003) *The Geography of Thought: How Asians and Westerners Think Differently … and Why.* New York, NY: The Free Press.

Riconscente, M. M. (2013) Results from a controlled study of the iPad fractions game Motion Math. *Games and Culture* **8**(4), 186-214.

Siegler R. S., Fazio L. K., Bailey D. H. & Zhou, X. (2013) Fractions: the new frontier for theories of numerical development. *Trends in Cognitive Sciences* **17**(1), 13-19.

Zorzi, M., Priftis, K. & Umiltà, C. (2002) Brain damage: neglect disrupts the mental number line. *Nature* **417**(6885), 138-139.

Zed: the structural link between mathematics and mathematics education

BHARATH SRIRAMAN

Zoltan Dienes (1916-2014), known as Zed, passed away on 11 January 2014. Some of us see it as a culmination of an era. Mathematics education is prone to forgetting its origins within the realm of mathematics, and Zed's passing away serves as a reminder to those who have witnessed the weakening of these origins. My intersection and interest with Zed's work began in the mid-1990s when there was excessive focus on the social turn in mathematics education. Being trained in mathematics, it was difficult to stomach at that point in time the associated set of sociological problems that were being addressed by mathematics education. I initiated a correspondence with Dienes which led to my discovery of *Building Up Mathematics* (Dienes, 1960) and *Thinking in Structures* (Dienes & Jeeves, 1965). Both these books have been influential to a generation of mathematics educators who entered the field in the 1970s and they remain classics to this day.

Trained as a mathematician in England, Zoltan became interested in the psychology of learning in the 1950s and earned a second degree in psychology. The field of mathematics education, seen through its origins in mathematics, is often outlined in terms of the classical tradition of Felix Klein followed by Freudenthal's re-conception with an emphasis on the humanistic element of doing mathematics. While the approach of Klein, steeped in an essentialist philosophy, gave way to the pragmatic approach of Freudenthal, Zoltan's approach influenced by structuralism and cognitive psychology remains unique from the point of view of developing a theory of learning which has left a lasting impact on the field. Most importantly this theory was grounded in fieldwork with school children that experienced the multi-embodiment approach to a mathematical idea through manipulatives, games, stories and even dance, before they were encouraged to abstract the essence of the activity leading to mathematical generalizations. The six-stages of learning consisted of free play, games, commonalities, representation, symbolization and finally formalization. I have always considered his approach to mathematical learning (and teaching) as falling within psychological structuralism à la Wilhelm Wundt because of its nuanced and layered approach to encouraging abstractions, with formalizations only occurring at the very end. This, to me, was similar to the focus on introspection as the method used by structuralists to understand conscious experience.

Zed's use of his theory of learning was powerful (to put it mildly). He had grown up surrounded by mathematicians. His father was a mathematician by training and gave Zed a book he authored on Taylor series for his 16th birthday. Zed's PhD thesis generalized one of Baire's category theorems by using Brower's intuitionist approach. In other words Zed believed in constructive mathematics in which *reductio ad absurdum* was viewed as a logical trick and frowned upon. When I met him in 2006, he was pushing 91, with a mind keen and fertile to talk mathematics (Sriraman & Lesh, 2007). I complained about being unable to find multiple embodiments to facilitate the learning of ideas in analysis. Two months later he sent me a paper on "A child's path to the Bolzano-Weierstrass theorem" (Sriraman, 2008), which essentially contained a structured story which allowed one to discover this deep theorem!

Zed embodied the common ground between mathematics and mathematics education, in a life that was dedicated to exploring the beauty of mathematics by making it accessible to schoolchildren. Given the climate of the "math wars" in the US and similar debates elsewhere in the world, it seems ironic that his seminal work on *Building Up Mathematics* remains forgotten. This book would appeal to both mathematicians and mathematics educators because of its focus on the foundational structures of mathematics.

Zed lived an adventurous life that included fieldwork spanning over 50 years with school children in the UK, Italy, Australia, Brazil, Canada, Papua New Guinea and the United States. His body of work will remain an inspiration for generations of mathematics educators who place mathematics at the center of mathematics education.

References

Dienes, Z. P. (1960) *Building Up Mathematics.* London, UK: Hutchinson Educational.

Dienes, Z. P. & Jeeves, M. A. (1965) *Thinking in Structures.* London, UK: Hutchinson Educational.

Sriraman, B. (Ed.) (2008) *Mathematics Education and the Legacy of Zoltan Paul Dienes.* Charlotte, NC: Information Age Publishing.

Sriraman, B. & Lesh, R. (2007) A conversation with Zoltan P. Dienes. *Mathematical Thinking and Learning* **9**(1), 59-75.

My students deserve better

PETER TAYLOR

In my third-year course, *Mathematical Explorations,* designed for future high school mathematics teachers, I had my students submit journals this year. Ranging from 4 to 40 pages, they discussed the problems we had worked on and reflected on their own learning. I see now that some of the real struggles they (and therefore I) seemed to be having, especially during the first half of the course, worked out pretty well for most of them. I also see that there are some things that might be changed. For example at the early stages it was hard to get much participation; I have to rethink my expectations.

My experience with this class has given me new insights into my two large first-year courses: calculus and linear algebra. I've been thinking about those courses over the past few years, trying different kinds of problems, different ways of interacting with the class, and though things seem to be working pretty well, I've always felt that there was something fundamentally amiss. My main purpose here is to think about ways in which those courses could be more like my third-year course.

In my third-year course, the problems we work with involve mathematics that most of the students have seen before but they are challenging in the sense that one has to play quite a bit in order to begin to see what sort of strategies might work. They are chosen for their power to deepen the students' understanding of the ideas and to lead them to a new appreciation of mathematical structure. According to the students, the main difference between this course and others they have taken lies in its pace (slower) and thrust (deeper and wider). (Aren't deeper and wider opposites? Not really—lateral connections reveal new structural properties.) The objective is as much to give the students a chance to confront and develop their learning skills as to deepen their mathematical understanding:

> As I reflect on my learning in math throughout my university career and in this course, I find that [...] I haven't "done" or "learned" math since high school; I have memorized and regurgitated the knowledge of my professors in hopes of getting good grades and finishing courses. The knowledge that I retained from all of this felt minimal, and it probably was, but this class helped me to do and learn math for real again. I realized that I did learn in my first three years in university but I didn't know how to apply my knowledge. Math became a daunting, scary mountain that I couldn't climb because I had forgotten how to apply what I know and really do math. But MATH 382 reminded me that math can be fun, and reminded me how to really DO math. (Kirsten)

What I discover from the students' journals is that this experience of digging deeply, of taking things apart to see how they work and then putting them back together, of constructing simple concrete examples as a way of playing with ideas, was new to most of the students. Remarkably enough, after 14 years of formal learning, they have spent almost no time in play.

That's not quite right. A number of our students, perhaps a quarter, have certainly spent a lot of time in their lives in mathematical play. When kids are young, they bend the rules and twist things into the wrong shape just to see what happens—that's their job as kids. But later on this natural behaviour seems to get schooled out of many of them, and they increasingly adopt safe strategies which seem to offer short-term gain. Only a few resist these temptations and keep right on playing. Who knows what makes the difference? Perhaps some early success, a key learning experience, an unusual teacher, or just a natural appetite for risk-taking. In any event such students do well in mathematics partly because they develop powerful learning strategies, but also simply because they've put in the time because they find playing with mathematics more fun than texting. I believe that our current undergraduate program serves these students very well.

It's the remaining, say, 75% of our students that I am interested in here. I have no doubt that these students have the capacity for serious play, but somehow, in their early years, they abandoned it, and it's hard for them to get started again. I know that there is considerable work being done on the question of how to get more students to keep on with that mathematical play. The question I am asking here, however, is: given the students I have now, what should I be doing in my large first-year classes?

> I had to think about it right then and there in the lecture when usually I'm just trying to keep up with the professor's handwriting, hardly listening to what they are saying. (Ashna)

The answer seems clear enough to me. I need to *teach less and discover more.* Rather than deliver the *product of mathematical thought,* engage them in the *process of mathematical thinking* (quoted from a paper by Asia Matthews). I don't mean to disparage "the product of mathematical thought" (more simply described as "mathematics"). It's real knowledge, particularly in a world in which much of what passes as knowledge is suspect. It's solid and eternal and has beauty and structure to die for. Nothing else in the knowledge world comes close to touching it. But my primary job as teacher is *not* to convey knowledge (narrowly interpreted), but to interpret, to transform, to enable, to bring to life.

I think that there are two components to this program. The first is to find a set of discovery problems that (if you like) cuts a natural path through the absurdly fat text-books that we far too often make our students buy in first year. And the second is to find a way to deliver those problems in a large class. I've been working on the first component for many years and it's coming along fine. The second component is more challenging and my experiments over the past years have had mixed success.

I don't use clickers, but I have often put out a small problem for the students to "pair and share" or simply think on their own. Sometimes this works quite well, but more often I feel, as I'm wandering around the room, that most of the students are sitting empty without much to think or say. When I start up again and ask for comments, the same few hands always go up. I'm thinking that this form of the consultative process doesn't work so well in mathematics.

> My first thought when Professor Taylor mentioned his struggles with his first-year students was that he definitely would have struggled to get me to engage when I was in first year […] if my professor asked me to collaborate with other students in class, I would probably just sit there and not contribute. (Michael)

Thinking back to my own student days I know that I would never have wanted to talk about a problem with a neighbour until I was ready to do so and that readiness can seldom be rushed. In fact I completely avoid discussing a problem until I have managed to centre it in my mind and assemble the necessary pieces beside it, and that takes time. As a student, what I wanted most from a lecture was a good story.

And that brings me to "discovery learning." Lately it's been in the news (not always favourably) and usually misunderstood. The most extreme misconception is that it expects students on their own to rediscover hundreds of years of hard-won knowledge. For me, discovery learning is best described as a style of communication. It begins with a problem or more generally with a narrative or story that is "open" in the sense that it invites exploration and further development. A lot of my curriculum work has involved the construction of such stories.

> I think my favourite part about this problem is the way it was framed; it makes it much more interesting and fun to solve. (Kirsten)

> I believe there is a lot of value in being able to work out a problem intuitively before exploiting any existing theorems or results, which seemed to be a theme that was emphasized throughout the course. (Makenna)

A story is a wonderful way of posing a dilemma, floating a paradox, setting up a quest. But then the action has to roll, the dilemma has to spin out and unwind. How is all that to happen in a large first-year class? Whitehead talked about this:

> In my own work at universities, I have been much struck by the paralysis of thought induced in pupils by the aimless accumulation of precise knowledge, inert and unutilized. It should be the chief aim of a university professor to exhibit himself in his own true character—that is, as an ignorant man, thinking, actively utilizing his small share of knowledge. (Whitehead, 1967/1929, p. 37)

We learn from example. A good example can be abstracted and retooled to fit onto a new problem. This applies also to learning how to learn. We learn about blocks and marbles by watching other kids play with them. We learn about playing with ideas by following the thoughts of a teacher.

> Other loosely related problems may have to be solved, to generate experience and insight. (Peiling)

In fact "playing with ideas" is not what it might seem. Ideas are abstract and play is concrete. We discover things by mucking about, by getting our hands around things, shapes, numbers, equations, concrete things as simple as we can make them without losing the piece of structure that has bedeviled us. Our first-year students can learn a lot simply by watching us reinvent examples, by witnessing that hands-on analogical process at work.

> This was my favorite problem because it shows math is very interesting and math is MAGIC. I cannot believe that math concept can help to construct such amazing pictures. The usual math problems I met are talking about proofs, derivatives and calculations. But this one can really trigger me to think something deeper—like math in my body. (Shuming)

So for me, discovery learning emerges when the student has wrestled with the problem in the tutorial or taken it home. But to enable that, to set it up, what's needed is what might be called discovery *teaching* and that's what Whitehead was describing: a reflective playing with an object of beauty.

> The amount of structure in this problem is truly amazing. (Jacob)

So that's my game plan for my first-year class this coming semester. Take a problem, a good problem with some marks of sophistication, and before their eyes, "actively utilizing my small share of knowledge," track it down, wrestle with it, bring it to the ground, and then stand back to let it rise up again, transparent with its inner structure displayed. Well, that's a plan; it puts a lot on the shoulders of first-year students. But if they are able to rise to the occasion, I will promise to organize the kind of technical and conceptual support they will need.

Postscript

I find a yearning for freedom in some of what my students have written: the freedom that comes from being in control and maybe even being a bit out of control. Anyway, the freedom of having your own hands on the wheel and your own foot on the accelerator. As a teacher, I also find myself looking for that kind of freedom and I know that other teachers do too. The hard thing about mathematics teaching, except at the advanced level, is that so much of the mathematics we teach is not the really the mathematics that we ourselves love and seek to spend time with. My students deserve better than that.

References

Whitehead, A. N. (1967/1929) *The Aims of Education and Other Essays.* New York, NY: Free Press.